Example

$D = \{\star\}$

Monoid theory:

(M1) $\forall x \forall y \forall z \, (x \star (y \star z) = (x \star y) \star z)$

(M2) $\exists e \forall x \, (e \star x = x \wedge x \star e = x)$

Prove: $\exists e \forall x \forall y \, (x \star (y \star e) = x \star y)$

Proof: By (M2), ~~[illegible]~~ we can write

$$\exists e \forall x \, (e \star x = x \wedge x \star e = x)$$

Remove $\exists$:

$$\forall x \, (e_0 \star x = x \wedge x \star e_0 = x)$$

Remove $\forall x$ – replace x with $x \star y$ to get

$$e_0 \star (x \star y) = x \star y \wedge (x \star y) \star e_0 = x \star y$$

By (M1),

$$\forall x \forall y \forall z \, (x \star (y \star z) = (x \star y) \star z)$$

Remove $\forall x$ – replace x with x

$\forall y$ – replace y with y

$\forall z$ – replace z with e_0,

$$x \star (y \star e_0) = (x \star y) \star e_0$$

Conclude by $\wedge$-elimination

$$\underbrace{(x \star y) \star e_0}_{\text{replace with } x \star (y \star e_0)} = x \star y$$

replace with $x \star (y \star e_0)$ to get

$$x \star (y \star e_0) = x \star y$$

Reintroduce $\forall x \forall y$:

$$\forall x \forall y \, (x \star (y \star e_0) = x \star y)$$

$\rightarrow$ Reintroduce $\exists$:

$$\exists e \forall x \forall y \, (x \star (y \star e) = x \star y)$$